Graph operations and synchronization of complex networks

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The effects of graph operations on the synchronization of coupled dynamical systems are studied. The operations range from addition or deletion of links to various ways of combining networks and generating larger networks from simpler ones. Methods from graph theory are used to calculate or estimate the eigenvalues of the Laplacian operator, which determine the synchronizability of continuous or discrete time dynamics evolving on the network. Results are applied to explain numerical observations on random, scale-free, and small-world networks. An interesting feature is that, when two networks are combined by adding links between them, the synchronizability of the resulting network may worsen as the synchronizability of the individual networks is improved. Similarly, adding links to a network may worsen its synchronizability, although it decreases the average distance in the graph.

I. INTRODUCTION

The synchronization of coupled systems is an active field of research with applications in many areas of the physical and biological sciences (see Ref. [1] for a general introduction). Synchronization is a wide-ranging phenomenon, which can be observed in systems ranging from pulse-coupled neurons [2] to chaotic oscillators [3], and even in the presence of delayed information transmission [4]. In some cases it is a desired phenomenon, such as when several lasers are coupled to obtain maximum power output, while in other cases it represents a pathology, such as the synchronous neural activity during epileptic seizures. The connection structure of a network plays an important effect on its synchronization. For diffusively coupled identical systems, the effects of the network topology can be expressed in terms of the spectrum of a diffusion (or Laplacian) operator (e.g., Refs. [5–9]). The spectrum characterizes synchronization for a given network structure; however, it usually gives little insight into the effects of changes in the structure. It is often difficult to say how some structural change might affect synchronization without calculating the eigenvalues afresh for the network.

In this paper we study the effects of changes in the network structure on the synchronization of coupled dynamical systems. The operations we consider include adding or removing links from the network, combining two or more networks into one, and generating large networks from simpler ones. Using ideas from graph theory, we deduce the spectrum of the resulting network from the spectrum of the original, without resorting to lengthy calculations. For certain operations the exact values of the eigenvalues can be obtained, while for others useful estimates are derived. Our results allow a systematic study of the synchronizability of whole classes of networks, and offer additional insight into the relation between network topology and synchronization.

The synchronization of coupled chaotic systems depends typically on a number of factors, including the strength of the coupling, the connection topology, and the dynamical characteristics of the individual units, quantified by, e.g., the maximal Lyapunov exponent. To introduce some notation, consider a network of identical systems, indexed by $i = 1, \ldots, n$, and governed by the differential equations

$$\dot{x}_i(t) = f(x_i(t)) - \kappa \sum_{j=1}^{n} L_{ij} x_j(t). \tag{1}$$

Here $\kappa \in \mathbf{R}$ denotes the coupling strength. The matrix of coupling coefficients $L = [L_{ij}]$ is symmetric, has the diagonal elements L_{ii} equal to the number of connections to unit i, and the off-diagonal elements L_{ij} are -1 if the ith and jth units are coupled and zero otherwise. A standard result is that the eigenvalues of L are real and non-negative, the smallest one is equal to zero, and is a simple eigenvalue if the network is connected. We assume a connected network and order the eigenvalues as $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$. For convenience we also use the notation $\lambda_{\min}$ and $\lambda_{\max}$ for the smallest and largest nonzero eigenvalues, λ_2 and λ_n, respectively. The system (1) is said to synchronize if $|x_i(t) - x_j(t)| \to 0$ as $t \to \infty$ for all i, j, starting from some open set of initial conditions. Synchronization may occur even when the individual dynamics are chaotic, and is related to the eigenvalues of L. For the system (1), the relevant condition is of the form [10]

$$\lambda_{\min} > \alpha, \tag{2}$$

where α is a quantity that depends on κ and f (more precisely, on the maximum Lyapunov exponent of f) but not on the connection topology. In another commonly studied system, namely the so-called coupled map lattice

$$x_i(t+1) = f(x_i(t)) - \kappa \sum_{j=1}^{n} L_{ij} f(x_j(t)), \tag{3}$$

the synchronization condition takes the form [9]

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$$\frac{\lambda_{min}}{\lambda_{max}} > \alpha. \tag{4}$$

We remark that (4) can also be the relevant condition for the synchronization of continuous-time systems if a more general coupling function is used than the one appearing in (1). Thus, synchronization depends on the eigenvalues λ_{min} and λ_{max} of L, a larger value of λ_{min} or a smaller value of λ_{max} indicating that synchronization can be achieved for a larger set of parameter values, e.g., for larger Lyapunov exponents or a wider range of coupling strengths.

The eigenvalues λ_i depend only on the connection structure and so can be introduced in a graph-theoretic way. Hence, consider the graph G underlying the coupled system. The vertices of G correspond to the dynamical units, with edges designating the interaction between them, which is assumed to be bidirectional, so G is an undirected graph. All graphs considered in this paper are assumed to be connected and simple (i.e., without loops or multiple edges). To avoid some trivial cases, we also assume throughout that they have at least two vertices. We write $V(G)$ and $E(G)$, respectively, for the vertex and edge sets of a given graph G. Conversely, we let $G=(E,V)$ denote the graph formed from given edge and vertex sets. The notation $|V(G)|$ denotes the number of vertices of G. The connection structure of G is described by its adjacency matrix $A=[a_{ij}]$ with elements $a_{ij}=1$ if the ith and jth vertices are connected by an edge, and zero otherwise. Let D be the diagonal matrix of vertex degrees, i.e., its ith diagonal element is the number of edges incident on the ith vertex. The coupling matrix of the dynamical system is then the Laplacian $L=D-A$ of its underlying graph. (A simple example is given in the Appendix.) We also write $L(G)$ and $\lambda_i(G)$ for the Laplacian and its eigenvalues when we wish to emphasize the dependence on the particular graph G. The smallest positive eigenvalue λ_{min} is called the algebraic connectivity of the graph [11] or the spectral gap of the Laplacian. By the condition (2), it also provides a measure by which various network architectures can be ranked with respect to the ease of synchronization of continuous-time systems (1) defined on them. Hence we can say that the graph G_1 is a better or poorer synchronizer than G_2 if $\lambda_{min}(G_1)$ is larger or smaller than $\lambda_{min}(G_2)$, respectively. Similarly, network architectures can also be compared with respect to the synchronizability condition (4) using the quantity $\lambda_{min}/\lambda_{max}$.

This correspondence between topology and dynamics allows the use of mathematical tools from graph theory for the investigation of synchronization, which we utilize in the following sections. Section II considers the Cartesian product, the join, and the coalescence operations, which can be viewed as specific ways of combining networks, as well as the more general operations of adding and removing links within a network or between two networks. For each operation, we determine the synchronizability of the resulting graph by calculating or estimating the quantities λ_{min} and $\lambda_{min}/\lambda_{max}$ from the eigenvalues of the original graph. Analytical predictions are numerically confirmed in Sec. III on random, scale-free, and small-world networks. In addition,
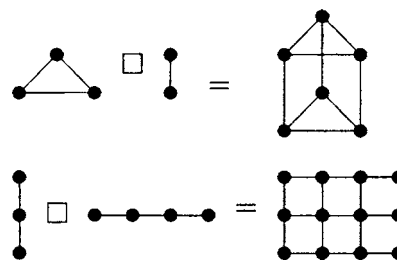

FIG. 1. The Cartesian product, $C_3 \square P_2$ (upper row) and $P_3 \square P_4$ (lower row).

several interesting features observed in the calculations are explained using the theory of Sec. II.

II. GRAPH OPERATIONS

In this section we consider several graph operations and their effects on the eigenvalues of the Laplacian. We start by recalling some elementary estimates on the eigenvalues. Since the diagonal elements of the Laplacian L are simply the vertex degrees d_i, it follows that

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} d_i. \tag{5}$$

Using the fact that $\lambda_1=0$, we have $\sum_{i=1}^{n}\lambda_i \leq (n-1)\lambda_{max}$. In other words,

$$\lambda_{max} \geq \frac{1}{n-1}\sum_{i=1}^{n} d_i > d_{avg}, \tag{6}$$

where d_{avg} denotes the average degree.

Another useful estimate for a graph on n vertices is [12]

$$\lambda_i \leq n \quad \text{for all } i. \tag{7}$$

A. Cartesian product

The Cartesian product is one of the basic operations on graphs, through which some common graphs can be constructed from simpler ones. For instance, regular grids, cubes, and their counterparts in higher dimensions are obtained from the Cartesian product of paths (linear chains) P_k. To give a precise definition, let $G=(V,E)$ and $H=(W,F)$ be two nonempty graphs. The *Cartesian product* $G\square H$ is a graph with vertex set $V \times W$, and $(x_1,x_2)(y_1,y_2)$ is an edge in $E(G\square H)$ if and only if either $x_2=y_2$ and $x_1y_1 \in E(G)$ or if $x_1=y_1$ and $x_2y_2 \in E(H)$. One may view $G\square H$ as the graph obtained from G by replacing each of its vertices with a copy of H and each of its edges with $|V(H)|$ edges joining corresponding vertices of H in the two copies. For instance, the product of two paths P_n and P_m yields an $n \times m$ rectangular grid. Some examples are given in Fig. 1.

The Cartesian product is a commutative, associative binary operation on graphs, see e.g., Ref. [13]. The eigenvalues for the product can be calculated from the eigenvalues for the factor graphs.

Proposition 1: The eigenvalues of the Laplacian for the

Cartesian product $G\square H$ satisfy

$$\lambda_{\min}(G\square H) = \min\{\lambda_{\min}(G), \lambda_{\min}(H)\},$$

$$\lambda_{\max}(G\square H) = \lambda_{\max}(G) + \lambda_{\max}(H),$$

$$\frac{\lambda_{\min}(G\square H)}{\lambda_{\max}(G\square H)} < \min\left\{\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}, \frac{\lambda_{\min}(H)}{\lambda_{\max}(H)}\right\}.$$

Proof: Suppose that G and H have s and r vertices, respectively. A result from graph theory implies that the eigenvalues of $L(G\square H)$ are given by all possible sums $\lambda_i(G) + \lambda_j(H)$, $1 \leq i \leq r$ and $1 \leq j \leq s$ (see, for example, Ref. [13]). Recalling that λ_1 is always zero, we have $\lambda_{\min}(G\square H) = \min\{\lambda_{\min}(G), \lambda_{\min}(H)\}$. It also follows that $\lambda_{\max}(G\square H) = \lambda_{\max}(G) + \lambda_{\max}(H)$. Finally, assume without loss of generality that $\lambda_{\min}(G) \leq \lambda_{\min}(H)$. Then

$$\frac{\lambda_{\max}(G\square H)}{\lambda_{\min}(G\square H)} = \frac{\lambda_{\max}(G) + \lambda_{\max}(H)}{\min\{\lambda_{\min}(G), \lambda_{\min}(H)\}} = \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)} + \frac{\lambda_{\max}(H)}{\lambda_{\min}(G)}$$

$$\geq \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)} + \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}$$

$$> \max\left\{\frac{\lambda_{\max}(G)}{\lambda_{\min}(G)}, \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}\right\},$$

which completes the proof. ∎

Thus, the product $G\square H$ cannot be a better synchronizer than its factors G and H, and in fact, with respect to the condition (4) it is a strictly poorer synchronizer than both G and H. Proposition 1 allows us to conclude simply by visual inspection that the graphs on the left-hand side of the equality signs in Fig. 1 are better synchronizers than those on the right-hand side. In particular, one-dimensional chains are better synchronizers than two-dimensional grids, which in turn are better than three-dimensional lattices, and so on.

Let us look more closely at the behavior of $\lambda_{\min}$ under the Cartesian product. The product $G\square P_2$ is two copies of G with edges between the corresponding vertices in each copy. The eigenvalues of $L(P_2)$ are 0 and 2, i.e., $\lambda_{\min}(P_2) = \lambda_{\max}(P_2) = 2$. Proposition 1 implies that

$$\lambda_{\min}(G\square P_2) = \begin{cases} \lambda_{\min}(G) & \text{if } \lambda_{\min}(G) \leq 2, \\ 2 & \text{if } \lambda_{\min}(G) > 2. \end{cases} \quad (8)$$

In other words, $\lambda_{\min}$ of the product graph saturates if $\lambda_{\min}(G)$ increases beyond 2. The more general product $G\square H$ is formed from several copies of G where corresponding vertices in each copy are connected according to the structure given by H, and $\lambda_{\min}(G\square H)$ is given by the right-hand side of (8) with two replaced by $\lambda_{\min}(H)$. One can fix H (and thus the connection structure of the copies of G), and study the resulting synchronizability for different choices of G. If $\lambda_{\min}(G)$ is increased (for instance, by adding edges within G, see Sec. II D), we see from the above Proposition that $\lambda_{\min}(G\square H)$ will increase as long as $\lambda_{\min}(G) < \lambda_{\min}(H)$, but not beyond the value $\lambda_{\min}(H)$. Thus the structure of H sets an upper bound on the synchronizability of $G\square H$, and this bound is quantified by $\lambda_{\min}(H)$. Since adding edges to G will not improve synchronization of the product graph beyond
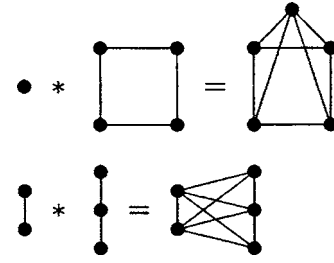

FIG. 2. The join operation, $P_1 * C_4$ (upper row) and $P_2 * P_3$ (lower row).

this limit, one might try adding links *across* the copies of G instead. For instance, instead of the product $G\square P_2$ where each vertex of G is coupled to its twin in the second copy, one can link each vertex of G to every vertex in the second copy of G. This is the *join* operation, which is the topic of Sec. II B. There it will be seen that $\lambda_{\min}$ can indeed be greatly increased by this procedure.

The situation is more interesting with respect to the ratio $\lambda_{\min}/\lambda_{\max}$. We show in Sec. III that, for fixed H, $\lambda_{\min}/\lambda_{\max}$ can actually *decrease* for $G\square H$ while $\lambda_{\min}(G)/\lambda_{\max}(G)$ *increases*. In other words, the better G synchronizes, the worse will the product $G\square H$. We will show in Sec. II E that this phenomenon is not peculiar to the Cartesian product, and can also occur under quite general connection of networks.

B. Join

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs on disjoint sets of n and m vertices, respectively. Their *disjoint union* $G_1 + G_2$ is the graph $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2)$, and their *join* $G_1 * G_2$ is the graph on $r = n + m$ vertices obtained from $G_1 + G_2$ by inserting edges from each vertex of G_1 to each vertex of G_2. (See Fig. 2)

Proposition 2: Let G and H be graphs on n and m vertices, respectively. Then the eigenvalues of the Laplacian for the join $G * H$ satisfy

$$\lambda_{\min}(G * H) = \min\{\lambda_{\min}(G) + m, \lambda_{\min}(H) + n\} \quad (9)$$

$$\geq \lambda_{\min}(G) + \lambda_{\min}(H) \quad (10)$$

and

$$\lambda_{\max}(G * H) = m + n. \quad (11)$$

If G and H have the same number of vertices, then

$$\frac{\lambda_{\min}(G * H)}{\lambda_{\max}(G * H)} > \frac{1}{2}. \quad (12)$$

Proof: The eigenvalues of $L(G * H)$ are given by $0, n + m, m + \lambda_i(G), 2 \leq i \leq n$; and $n + \lambda_j(H), 2 \leq j \leq m$ [13], from which (9) and (11) follow. By (7), $m \geq \lambda_{\min}(H)$ and $n \geq \lambda_{\min}(G)$, which imply (10). It follows that when $n = m$,

$$\frac{\lambda_{\min}(G * H)}{\lambda_{\max}(G * H)} = \frac{n + \min\{\lambda_{\min}(G), \lambda_{\min}(H)\}}{2n} > \frac{1}{2}.$$
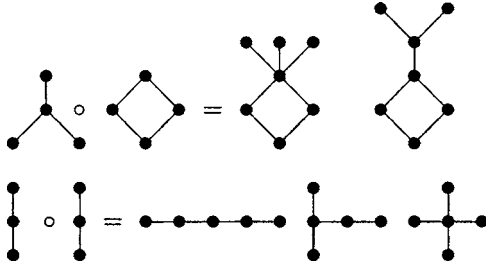
∎

FIG. 3. All coalescences of a star with four vertices and the cycle C_4 (upper row), and all coalescences of P_3 and P_3 (lower row).

Hence, $\lambda_{\min}(G*H)$ is always larger than both $\lambda_{\min}(G)$ and $\lambda_{\min}(H)$, and therefore also larger than $\lambda_{\min}(G\square H)$, by Proposition 1. The situation is not so clear cut for the ratio $\lambda_{\min}/\lambda_{\max}$. For example, for the special case $G*G$, $\lambda_{\min}/\lambda_{\max}$ is usually higher for the join than for the original graph G, but it can also be smaller. In order to have $\lambda_{\min}(G*G)/\lambda_{\max}(G*G) \le \lambda_{\min}(G)/\lambda_{\max}(G)$, a necessary condition from (12) is that

$$\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)} > \frac{1}{2}, \tag{13}$$

that is, G should already be a good synchronizer. Figure 4 in Sec. III gives an example for the case when (13) is satisfied and $G*G$ is a slightly poorer synchronizer than G. Nevertheless, by (12) the join of two graphs of comparable sizes will yield a good synchronizer, and for specific graph types it can be proved that the result will be strictly better than the individual graphs. We give an example for the case of trees.

Corollary 1: Let T and S be two trees each having n vertices, then

$$\frac{\lambda_{\min}(T*S)}{\lambda_{\max}(T*S)} > \max\left\{\frac{\lambda_{\min}(T)}{\lambda_{\max}(T)}, \frac{\lambda_{\min}(S)}{\lambda_{\max}(S)}\right\}.$$

Proof: A tree T with n vertices has $n-1$ edges, so the sum of its vertex degrees is $2(n-1)$, and by (6) $\lambda_{\max}(T) \ge 2$. Furthermore, it is a known fact that $\lambda_{\min}(T) \le 1$ for trees [11]. Thus, $\lambda_{\min}(T)/\lambda_{\max}(T) \le 1/2$. The corollary then follows by (12). ∎

C. Coalescence

A *coalescence* of G_1 and G_2 is any graph obtained from the disjoint union G_1+G_2 by identifying a vertex of G_1 with a vertex of G_2, i.e., merging one vertex from each graph into a single vertex. Unlike the Cartesian product, the coalescence generally does not yield a unique graph; see Fig. 3. We denote by $G_1 \bigcirc G_2$ any coalescence of G_1 and G_2. We show that $G \bigcirc H$ cannot be a better synchronizer than G or H. Note that the result holds for *all* possible coalescences of the pair G,H, and thus applies to a large number of graph types.

Proposition 3: For any coalescence $G \bigcirc H$ of G and H,

$$\lambda_{\min}(G \bigcirc H) \le \min\{\lambda_{\min}(G), \lambda_{\min}(H)\},$$

$$\lambda_{\max}(G \bigcirc H) \ge \max\{\lambda_{\max}(G), \lambda_{\max}(H)\},$$

$$\frac{\lambda_{\min}(G \bigcirc H)}{\lambda_{\max}(G \bigcirc H)} \le \min\left\{\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}, \frac{\lambda_{\min}(H)}{\lambda_{\max}(H)}\right\}.$$

Proof: Suppose that $b=(b_1,\ldots,b_p)$ and $c=(c_1,\ldots,c_q)$ are sequences of non-negative real numbers arranged in nonincreasing order. We say that b majorizes c if $\Sigma_{i=1}^k b_i \ge \Sigma_{i=1}^k c_i, 1 \le k \le \min(p,q)$, and $\Sigma_{i=1}^p b_i = \Sigma_{i=1}^q c_i$. Let Spec $(G) = [\lambda_{\max}(G),\ldots,\lambda_{\min}(G),0]$ be the sequence of eigenvalues of $L(G)$ in nonincreasing order. By a result of Grone and Merris [14], Spec $(G \bigcirc H)$ majorizes Spec $(G+H)$ for any coalescence $G \bigcirc H$ of G and H. By majorization the result is proved. ∎

Applying Proposition 3 to the graphs in Fig. 3, we can immediately see without any calculations that the star S_5 on 5 vertices cannot be a better synchronizer than the linear chain P_3. Indeed, the ratio $\lambda_{\min}/\lambda_{\max}$ is $1/5$ for S_5 and $1/3$ for P_3. Similarly, shorter chains will synchronize better than longer ones, since the latter can be viewed as a coalescence of the former. Similar conclusions can be drawn for the other graphs by visual inspection. The numerical calculations in Sec. III confirm that the coalescence operation yields reduced synchronizability, often by an order of magnitude.

D. Adding and removing edges

If $G=(V,E)$ is a graph, then $G-e$ denotes the graph obtained from G by removing the edge $e \in E(G)$. If $e \notin E(G)$, then $G+e$ is the graph obtained from G by adding an edge e. In general, the precise effect on the spectrum of adding or deleting edges is still poorly understood. One well-known fact is [12]

$$\lambda_i(G+e) \ge \lambda_i(G), \quad 1 \le i \le n.$$

In particular, $\lambda_{\min}(G+e) \ge \lambda_{\min}(G)$, so when edges are added (respectively, deleted), the synchronizability of the system (1) either increases (respectively, decreases) or stays the same. On the other hand, it is not easy to say what will happen to the ratio $\lambda_{\min}/\lambda_{\max}$, so a similar conclusion cannot be drawn for the coupled map lattice (3). The computations in Sec. III verify that, for a fixed number of vertices, $\lambda_{\min}$ is nondecreasing as the number of edges is increased. However, there are cases where $\lambda_{\min}/\lambda_{\max}$ can strictly decrease with increasing edge number. We study such a case in more detail in the next section, where we consider the arbitrary connection of two networks.

E. Connecting two networks

We now consider connecting two separate networks by adding links between them. The next result gives estimates for the synchronizability of the resulting network.

Proposition 4: Let G_1, G_2 be two graphs on n_1 and n_2 vertices, respectively, and let H be the graph obtained by adding k edges between G_1 and G_2. Then,

$$\lambda_{\min}(H) \le \frac{2k}{\min\{n_1,n_2\}}, \tag{14}$$

$$\frac{\lambda_{\min}(H)}{\lambda_{\max}(H)} < \frac{2k}{d_{\text{avg}}(H)\min\{n_1,n_2\}}. \tag{15}$$
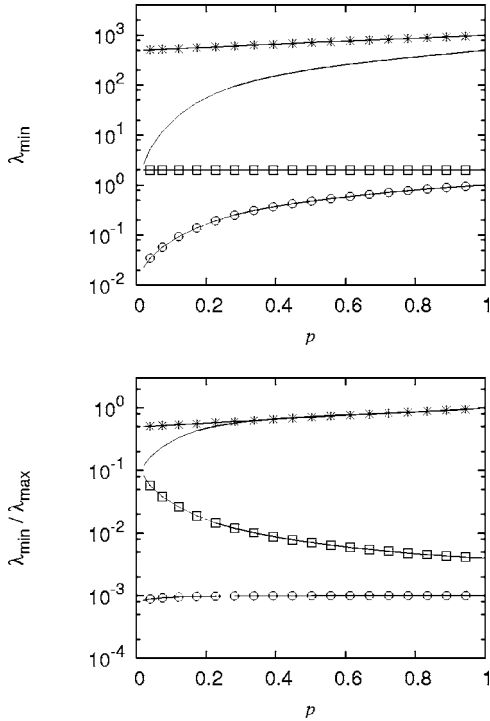
FIG. 4. Synchronizability of random networks under graph operations. The original graph G is shown with the solid line (—), while the other curves correspond to the join $G*G$(-*-), the Cartesian product $G\square P_2$ (-$\square$-), and the coalescence $G\bigcirc G$ (-$\bigcirc$-).

Proof: The proof makes use of the notion of the isoperimetric number of a graph. For a subset X of $V(G)$, we define the boundary ∂X as the set of edges $e = xv$ such that $x \in X$ and $v \notin X$. We let $|S|$ denote the cardinality of a set S. The *isoperimetric number* of G is defined as $i(G) = \min_{X \subset V(G)} |\partial X|/|X|$, where the minimum is taken over all subsets X of the vertex set of G satisfying $1 \leq |X| \leq |V(G)|/2$. The quantities $\lambda_{\min}(G)$ and $i(G)$ are related by [15]

$$\lambda_{\min}(G) \leq 2i(G). \qquad (16)$$

Hence, for the graph H, we have

$$\lambda_{\min}(H) \leq 2i(H) \leq 2\frac{|\partial X|}{|X|},$$

where X is any subset of $V(H)$ satisfying $1 \leq |X| \leq |V(G)|/2$. In particular, X can be chosen as the smaller of G_1 and G_2, which yields (14). Using (6), we obtain (15). ∎

Suppose now that two graphs G_1, G_2 are combined as in the above proposition, and consider adding edges to G_1 or G_2 while keeping constant the number of connections k between them. The effect is to increase the average degree $d_{\mathrm{avg}}(H)$ of the combined graph and thus decrease the right-hand side of (15). Since (15) is only an upper bound, the net effect on $\lambda_{\min}(H)/\lambda_{\max}(H)$ cannot be determined for small values of $d_{\mathrm{avg}}(H)$. However, when $d_{\mathrm{avg}}(H)/k$ is made sufficiently large by adding enough edges, then (15) implies that $\lambda_{\min}(H)/\lambda_{\max}(H)$ should be small. Therefore, adding more links in a network may impede the synchronization, although
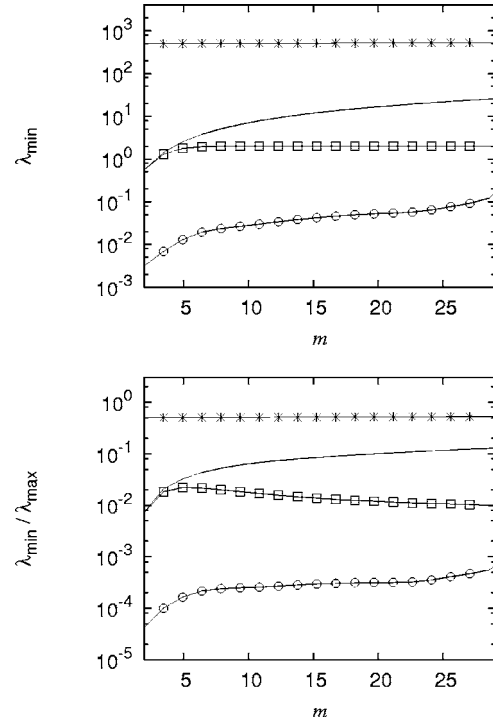

FIG. 5. Synchronizability of scale-free networks under graph operations. The curves correspond to G(—), $G*G$(-*-), $G\square P_2$(-$\square$-), and $G\bigcirc G$(-$\bigcirc$-).

it decreases the average distance and the diameter. Figure 7 in the next section numerically confirms this observation.

III. NUMERICAL RESULTS

We now use numerical calculations to obtain more detailed information on the effects of the graph operations presented in Sec. II. Our aim is to determine the behavior of several common architectures, namely random, scale-free, and small-world networks, in relation to the graph operations. Furthermore, we wish to explain the numerical observations using the foregoing theoretical considerations.

The calculations in this section are done on networks of 500 vertices and by averaging the results over several realizations. The random networks are constructed starting with a fixed number of vertices and adding an edge between any pair of vertices with probability p [16]. The scale-free networks are constructed using the Barabasi-Albert algorithm for preferential attachment [17], starting with m initial vertices, and adding a vertex at each step with m links to existing vertices with probability proportional to their vertex degrees. For small-world networks we use the variant proposed in Ref. [18], which is obtained by randomly adding m links to a cycle (a set of vertices connected to their nearest neighbors in circular arrangement). The results are summarized in Figs. 4–6. In each figure, the synchronizability of the graph G is compared to that of the join $G*G$, the Cartesian product $G\square P_2$, and the coalescence $G\bigcirc G$. The graph $G\bigcirc G$ is constructed by taking two copies of G and coalescing a randomly selected vertex from each copy.
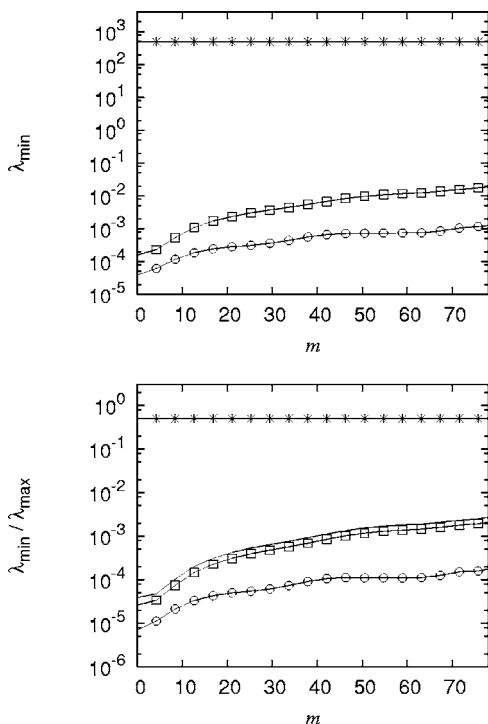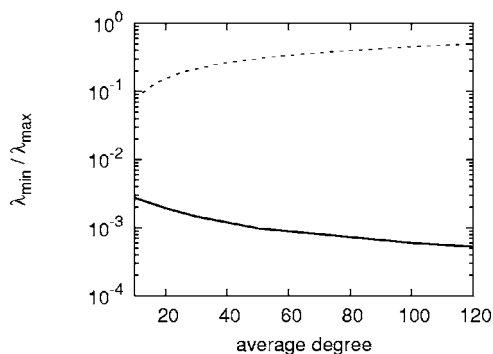
FIG. 7. The lower solid curve is the ratio $\lambda_{min}/\lambda_{max}$ for two copies of a random graph G of 500 vertices connected to each other by 20 links. The upper dotted curve gives the same ratio for the single graph G.

FIG. 6. Synchronizability of small-world networks under graph operations. The curves correspond to $G(—), G*G(\text{-}*\text{-})$, $G\Box P_2(\text{-}\Box\text{-})$, and $G \bigcirc G(\text{-}\bigcirc\text{-})$. In the upper figure, the solid line corresponding to G is indistinguishable since $\lambda_{min}(G)$ coincides with $\lambda_{min}(G\Box P_2)$).

Some general features of the graph operations are apparent from the Figs. 4–6. For instance, the join $G*G$ typically yields improved synchronizability compared to the original network G, whereas the Cartesian product $G\Box P_2$ results in reduced synchronizability, and the coalescence $G\bigcirc G$ has the worst synchronizability, as measured by λ_{min} or $\lambda_{min}/\lambda_{max}$. An exception is the random graph where $\lambda_{min}/\lambda_{max}$ for $G*G$ is about the same as (and can in fact be slightly less than) the corresponding ratio for G when G itself is a sufficiently good synchronizer. Note that this happens when $\lambda_{min}(G)/\lambda_{max}(G)$ becomes larger than $1/2$, in agreement with (13). As the vertical scales in the figures are logarithmic, the synchronizability of the networks can often differ by several orders of magnitude after the graph operations.

We now use the results of the preceding sections to give a theoretical understanding of several interesting observations from the figures. We first note that the value of λ_{min} for the Cartesian product $G\Box P_2$ saturates to 2 in the random and scale-free networks of Figs. 4 and 5. This is a consequence of (8) and the fact that $\lambda_{min}(G)$ increases as more edges are added to G (shown by the solid curve in the figures). On the other hand, for the small-world network of Fig. 6, the curve $\lambda_{min}(G\Box P_2)$ coincides with $\lambda_{min}(G)$, in agreement with (8) since $\lambda_{min}(G)$ is less than 2 in this case.

With respect to the join operation, we have $\lambda_{min}(G*G) = 500 + \lambda_{min}(G)$, as predicted by (9) and verified by the figures. In the scale-free and small-world networks $\lambda_{min}(G*G)$ appears almost as a horizontal line at the value 500 since $\lambda_{min}(G)$ is relatively small in these cases.

We also note that in most cases synchronizability increases as edges are added (i.e., as p or m is increased). However, a notable exception occurs for the Cartesian product $G\Box P_2$ of random and scale-free networks. Here, over a large interval the ratio $\lambda_{min}/\lambda_{max}$ monotonically decreases for $G\Box P_2$, although it increases for G, as more edges are added. In other words, the better the synchronizability of G, the worse is the synchronizability of $G\Box P_2$. This interesting situation is most prominent in purely random networks (Fig. 4), somewhat less conspicuous in scale-free networks (Fig. 5), and absent in small-world networks (Fig. 6). We use the theory of Sec. II to give a quantitative account. Thus, by Proposition 1, Eq. (8), and the fact that $\lambda_{max}(P_2)=2$, we have

$$\frac{\lambda_{min}(G\Box P_2)}{\lambda_{max}(G\Box P_2)} = \frac{2}{2 + \lambda_{max}(G)} \quad \text{if } \lambda_{min}(G) \geqslant 2.$$

As edges are added to G, $\lambda_{max}(G)$ will generally increase as mentioned in Sec. II D, so $\lambda_{min}/\lambda_{max}$ will decrease over the range where $\lambda_{min}(G) \geqslant 2$. This latter inequality holds for the random graph of Fig. 4 throughout the range of parameters used, so $\lambda_{min}/\lambda_{max}$ decreases monotonically, while the opposite is true for the small-world network of Fig. 6. Furthermore, for the scale-free network of Fig. 5, $\lambda_{min}/\lambda_{max}$ initially increases, but begins to decrease at the point where $\lambda_{min}(G)$ reaches 2.

The phenomenon of decreased synchronizability with increased connectivity can also be observed in the general coupling of two networks. Indeed, if two copies of a graph G having n vertices are connected by adding k links between them, then for the resulting graph H we estimate

$$\frac{\lambda_{min}(H)}{\lambda_{max}(H)} < \frac{2k}{nd_{avg}(H)} \tag{17}$$

using (15). If k is small compared to the size of G, then $nd_{avg}(H) \approx nd_{avg}(G)$, i.e., about twice the number of edges of G. Thus, the right-hand side of (17) can be viewed as the ratio of the number of links between the two copies of G to the number of links within G. By increasing the average degree within G, $\lambda_{min}(H)/\lambda_{max}(H)$ can be made smaller. Figure 7 shows that $\lambda_{min}/\lambda_{max}$ decreases for the combined graph H while it increases for the individual graph G. By a straight-

forward extension of this argument, a similar conclusion can be drawn when two different graphs G_1 and G_2 are connected by adding links between them.

IV. CONCLUSION

We have considered some common operations on graphs and studied their effects on the synchronization of coupled dynamical systems. For the Cartesian product and the join operations, the eigenvalues of the Laplacian for the resulting graph can be directly determined from those for the original graphs, which gives a method for determining synchronizability without lengthy calculations. For the other operations the eigenvalues can be estimated, providing useful insight into the relation between synchronization and connection topology. In simpler cases, the results allow one to determine which network is a better synchronizer simply by visual inspection of its structure. Such heuristics should be useful in design procedures.

We have illustrated our results numerically on random, scale-free, and small-world networks, and used the theory to explain several features observed in numerical calculations. For instance, we have shown that adding links to a graph may improve, saturate, or worsen its synchronizability, although the average distance of the graph decreases. A related observation is that, when two networks are combined by adding links between them, the synchronizability of the resulting network can worsen as that of the individual networks is improved. Using the theoretical results we are able to explain and predict when such situations can arise. Clearly, such analytical tools can help us better understand the structure and dynamics of complex networks. For example, it has recently been shown that the degree distribution of a network generally does not determine its synchronizability [19], a significant fact which is difficult to establish on the basis of numerical simulations alone. Several of the ideas presented here can be extended to more general coupling operators, such as weighted connection matrices. These results will be reported in a future work.

APPENDIX

The following is a short example illustrating the notation and the relevant matrices for a simple graph, namely a linear chain of three vertices, denoted by P_3 and depicted in the second row of Fig. 3. Labeling the vertices linearly in an obvious way, the vertex degrees (number of neighbors of each vertex) are 1, 2, and 1, which form the diagonal entries of the matrix $D = \text{diag}\{1,2,1\}$. The neighborhood relation of the graph is contained in the adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

and the Laplacian is given by

$$L = D - A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

The eigenvalues of L are 0, 1, and 3. Disregarding the trivial eigenvalue, we have $\lambda_{min} = 1$ and $\lambda_{max} = 3$ in our notation.

Using the graph operations on P_3, one obtains several graphs whose synchronizability can be ranked by the ratio $\lambda_{min}/\lambda_{max}$ of their respective Laplacians. The Cartesian product $P_3 \square P_3$ is a 3×3 rectangular grid with $\lambda_{min}/\lambda_{max} = 1/6$, so is a poorer synchronizer than P_3. The join $P_3 * P_3$ has $\lambda_{min}/\lambda_{max} = 4/6$, implying improved synchronizability over P_3. These values can be directly found from Propositions 1 and 2. All coalescences of $P_3 \bigcirc P_3$ are shown in the second row of Fig. 3, and for these graphs the ratio $\lambda_{min}/\lambda_{max}$ takes the values 0.106, 0.124, and 0.2 (from left to right), all showing decreased synchronizability over P_3. On the other hand, adding an edge to P_3 gives the cycle C_3, depicted in the first row of Fig. 1, which is a complete graph and has $\lambda_{min}/\lambda_{max} = 3/3$, the maximum possible value for any graph.

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